ing: "In the liguric documents of the second half of the fifteenth century we found in frequent use, to indicate the multiplication by 1000, in place of M, an O crossed by a horizontal line." This closely resembles some forms of our Spanish symbol U. Cappelli gives two facsimile reproductions[1] in which the sign in question is small and is placed in the position of an exponent to the letters XL, to represent the number 40,000. This corresponds to the use of a small *c* which has been found written to the right of and above the letters XI, to signify 1100. It follows, therefore, that the modified U was in use during the fifteenth century in Italy, as well as in Spain, though it is not known which country had the priority.

What is the origin of this *calderón*? Our studies along this line make it almost certain that it is a modification of one of the Roman symbols for 1000. Besides M, the Romans used for 1000 the symbols CIↃ, T, ∞ and ᛏ. These symbols are found also in Spanish MSS. It is easy to see how in the hands of successive generations of amanuenses, some of these might assume the forms of the calderón. If the lower parts of the parentheses in the forms CIↃ or CIIↃ are united, we have a close imitation of the U, crossed by one or by two bars.

Allied to the distorted Spanish U is the Portuguese symbol for 1000, called the "Cifrão."[2] It looks somewhat like our modern dollar mark $. But its function in writing numbers was identical with that of the calderón. Moreover, we have seen forms of this Spanish "thousand" which need only to be turned through a right angle to appear like the Portuguese symbol for 1000. Changes of that sort are not unknown. For instance, the Arabic numeral 5 appears upside down in some Spanish books and manuscripts as late as the eighteenth and nineteenth centuries.

A GENERAL CONSTRUCTION FOR CIRCULAR CUBICS.

By R. M. MATHEWS, Wesleyan University.

1. It is well known that any cubic curve can be generated as the locus of the intersections of a pencil of conics on four points with a projective pencil of lines. In practical work it is difficult to draw the conics and to effect the correlation involved in this construction. As a circular cubic passes through the circular points at infinity, the conics for such a curve may be specialized to a pencil of circles on two finite points. It remains, then, to find a simple and general method for effecting the correlation with the pencil of lines.

Schroeter and Durège in simultaneous papers[1] have shown that if each line pass through the center of the corresponding circle, the locus contains its singular

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[1] Adriano Cappelli, *op. cit.*, p. 436, first column, Nos. 5 and 6.

[2] See the word "Cifrão" in Antonio de Moraes Silva, *Dicc. de Lingua Portuguesa*, 1877; in Vieira, *Grande Dicc. Portuguez*, 1873; in *Dicc. Comtemp. da Lingua Portuguesa*, 1881.

[1] H. Schröter, "Ueber eine besondere Curve 3^{ter} Ordnung und eine einfache Erzeugungsart der allgemeinen Curve 3^{ter} Ordnung," *Mathematische Annalen*, vol. 5, 1872, pp. 50–82.

H. Durège, "Ueber die Curve 3^{ter} Ordnung, welche den geometrischen Ort der Brennpunkte einer Kegelschnittschaar bildet," *ibid.*, 83–94.

focus, which is the intersection of the tangents at the circular points. Moreover, the general construction has been considered for special base points[1]. This paper explains a general and simple compass and ruler construction for the case of an arbitrary pair of base points.

2. Let Q and R be two fixed points on a variable circle which is cut again in S and T by two arbitrary fixed lines s and t through Q and R, respectively. The variable line ST cuts an arbitrary fixed line g in U. The variable line PU, drawn from a fixed point P, cuts the circle in A and B. The locus of A and B for the pencil of circles through Q and R is a cubic curve. We find the following properties of the figure.

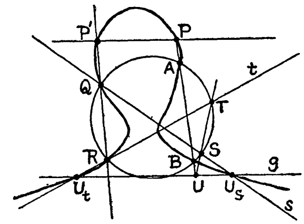

FIG. 1.

1. The lines ST are parallel, for $\angle\ TSQ = \angle\ QRT$ = constant.

2. There is a one-to-one correspondence between the circles and the pencil at P. For each circle cuts s and t in points S and T; the line ST determines U on g and so PU. Conversely, each line from P cuts g in a point U which determines one line in the fixed direction ST and so two points S and T concyclic with Q and R.

3. Q and R are on the locus. For PQ cuts g at U_Q and so the corresponding circle is determined; a similar consideration applies to R.

4. The circles of the pencil pass through the circular points at infinity, and thus the isotropic lines through P place them on the locus, which is, then, a circular cubic.

5. The circle through P puts this point on the curve.

6. When U is at infinity on g, then S and T are at infinity, and the circle on Q and R is replaced by the line QR and the line at infinity. Let the line through P parallel to g cut QR at P' and the line at infinity at F_∞, which is also on g.

7. The line g cuts the locus at F_∞ and at the points U_s and U_t where it meets s and t, respectively.

3. The given construction can now be shown applicable to any circular cubic. A circle through three points Q, R and A of the curve will cut it again in a finite point B. The line QR cuts the curve again in P' while AB does so in P. Then it is well known that PP' is parallel to the real asymptote. Let g, an arbitrary parallel to PP', cut the cubic in U_s and U_t. Thus lines s and t are determined as QU_s and RU_t, respectively. Our construction applied now on this basis will give a circular cubic which has in common with the given cubic the nine points, Q, R, P', A, B, P, F_∞, U_s and U_t besides the circular points at infinity. Accordingly, the cubics are identical.

4. The construction which Teixeira[2] derived analytically may be shown to be a special case of the foregoing. Take g through R and let it cut the cubic

[1] G. Loria, *Spezielle algebraische und transzendente ebene Kurven.* Leipzig, 1910, vol. 1, pp. 34–35.

[2] F. Gomes Teixeira, "Sur une manière de construire les cubiques circulaires," *Nouvelles Annales de Mathématiques*, fourth series, vol. 16, 1916, pp. 449–454.

again at U_s. Let the variable circle cut g again at H. Then as triangles QRU_s and SHU_s are similar:

$$\frac{U_sH}{U_sS} = \frac{U_sQ}{U_sR} = k, \text{ constant.}$$

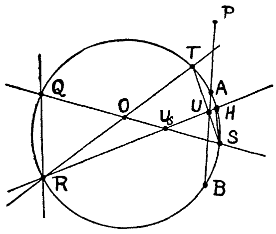

FIG. 2.

As the variable line ST moves parallel to itself, the segments it cuts on the fixed lines from U_s are proportional, so

$$\frac{U_sU}{U_sS} = k', \text{ constant.}$$

Thence

$$\frac{U_sH}{U_sU} = \frac{k}{k'} = c, \text{ constant.}$$

The procedure given is to draw PU arbitrarily from P to cut g at U; to determine H on g from the constant ratio just proved; and then to construct the circle through QRH to cut PU in points of the curve.

A GENERALIZATION OF THE STROPHOID.

By J. H. WEAVER, Ohio State University.

W. W. Johnson has given the following generalization for the strophoid.[1] Let A and B be two fixed points, and let two variable lines PA and PB make with AB angles ϕ and ψ, respectively. Let α be a constant angle and let P move so that

$$n\phi \pm m\psi = \alpha. \tag{1}$$

Then the locus of P is a strophoid. Equation (1) shows that there is associated with this set of curves a circle having a segment with AB as base in which the angle α may be inscribed.

In the following discussion some curves are developed which have associated with them the three conic sections.

Elliptic Case. Let there be an ellipse E with major axis AB, and from A and B let variable lines AP and BP be drawn making angles θ_1 and θ_2 respectively with AB. Let AQ and BQ be so drawn as to make angles $\pm m\theta_1$ and $\pm n\theta_2$ with AB. When the locus of Q is the ellipse E, the locus of P is a curve whose equation may be developed as follows. (In this development we will consider m and n as positive integers and relatively prime to each other.)

The slope of AQ is $\tan (m\theta_1)$ and of BQ is $\tan (n\theta_2)$. Then since Q is on E we have

$$\tan (m\theta_1) \cdot \tan (n\theta_2) = - b^2/a^2, \tag{2}$$

[1] "The Strophoids," *American Journal of Mathematics*, vol. 3, 1880, pp. 320–325. See also G. Loria, *Spezielle algebraische und transcendente ebene Kurven*, Berlin, vol. 1, 1910, p. 73. This lcass of curves includes the sextrix curves as a subclass. See Loria, l.c., p. 390.